Integral Theorems:

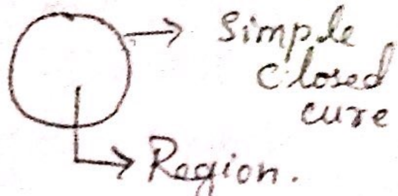
Introduction: Double integrals over a plane region may be transformed in to line integrals over the boundary of the region and conversely. This is of practical interest because it may help to make the evaluation of an integral easier. It also helps in the theory whenever one wants to switch from one kind of integral to the other. The transformation can be done by the following theorem.

Green's theorem in plane:

Definitions:

1) Simple Closed Curve: A closed curve which doesnot cross itself is called a simple closed curve. Circle, Ellipse, triangle, rectangle are examples.

2) Simply Connected region: A region bounded by a simple closed curve is called a simply Connected region.

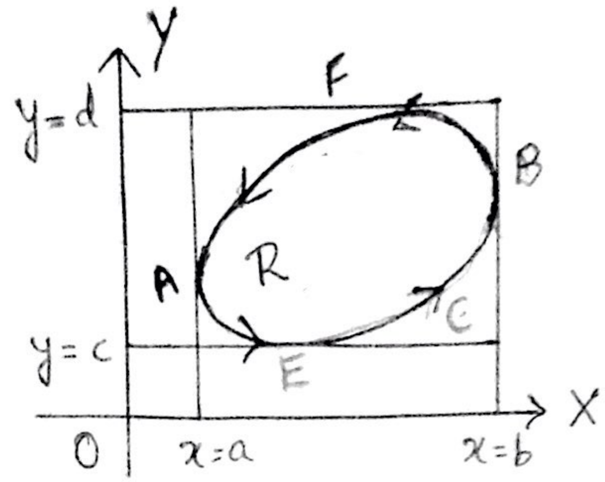


3) Jordon Curves: Simple closed curves that lie in a plane are usually called Jordon curves.

State and Prove Green's Theorem in the plane:

Statement: Let $P(x,y)$ and $Q(x,y)$ be two continuous functions having continuous partial derivatives in a region R of the xy-plane bounded by a simple closed curve C, then

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$



Proof: Assume that the curve C is positively oriented and can be divided into two parts.
Divide the curve C into two arcs AEB and BFA as shown in the figure.
Let $y = f_1(x)$ be the equation of the arc AEB and $y = f_2(x)$ be the equation of the arc AFB. Then,

$$\iint_R \frac{\partial P}{\partial y}\, dx\, dy = \int_{x=a}^{x=b} \left[\int_{y=f_1(x)}^{y=f_2(x)} \frac{\partial P}{\partial y}\, dy \right] dx$$

$$= \int_a^b \Big[P(x,y) \Big]_{f_1(x)}^{f_2(x)} dx$$

$$= \int_a^b \Big[P(x, f_2(x)) - P(x, f_1(x)) \Big] dx$$

$$= -\left\{ \int_a^b P(x, f_1(x))\, dx - \int_a^b P(x, f_2(x))\, dx \right\}$$

$$= -\left\{ \int_a^b P(x, f_1(x))\, dx + \int_b^a P(x, f_2(x))\, dx \right\}$$

$$= -\left\{ \int_{AEB} P(x, y)\, dx + \int_{BFA} P(x, y)\, dx \right\}$$

$$= -\oint_C P(x, y)\, dx \Rightarrow \oint_C P\,dx = -\iint_R \frac{\partial P}{\partial y}\, dx\,dy \longrightarrow (1)$$

Similarly dividing C in to arcs FAE and EBF. and its equations are $x = g_1(y)$ and $x = g_2(y)$.

$$\iint_R \frac{\partial Q}{\partial x}\, dx\,dy = \int_{y=c}^{y=d} \left[\int_{x=g_1(y)}^{x=g_2(y)} \frac{\partial Q}{\partial x}\, dx \right] dy$$

$$= \int_c^d \Big[Q(x,y) \Big]_{g_1(y)}^{g_2(y)} dy$$

$$= \int_c^d \Big[Q(g_2(y), y) - Q(g_1(y), y) \Big] dy$$

$$= \int_c^d Q(g_2(y), y)\, dy - \int_c^d Q(g_1(y), y)\, dy$$

$$= \int_c^d Q(g_2(y), y)\, dy + \int_d^c Q(g_1(y), y)\, dy$$

$$= \int_{FAE} Q(x,y)\, dy + \int_{EBF} Q(x,y)\, dy = \oint_C Q\, dy$$

$$\therefore \oint_C Q\, dy = \iint_R \frac{\partial Q}{\partial x}\, dx\, dy \longrightarrow (2)$$

from (1) & (2) $\displaystyle \oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$

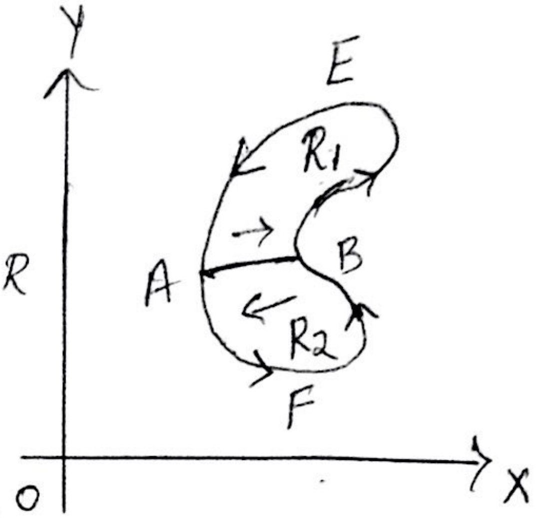
Hence the theorem.

Green's Theorem in vector form:

If $\vec{F} = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$

Then $$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\nabla \times \vec{F})\cdot \hat{k}\, dR \quad [dR = dx\,dy]$$

Extension of Green's theorem:



The Green's theorem can now be extended to a plane region R which can be divided in to finite number of sub regions, say R_1 and R_2 as shown in the figure by drawing a line AB. In this case we apply the theorem to each sub region R_1 and R_2 separately and then add the results.

$$\int_{ABEA} P\,dx + Q\,dy = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \longrightarrow (1)$$

$$\int_{AFBA} Pdx + Q\,dy = \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \rightarrow (2)$$

Now Add (1) and (2) of LHS

$$\int_{ABEA} Pdx + Qdy + \int_{AFBA} Pdx + Qdy = \int_{AB} + \int_{BEA} + \int_{AFB} + \int_{BA}$$

$$= \int_{AB} + \int_{BEA} + \int_{AFB} - \int_{AB}$$

$$= \int_{BEAFB} Pdx + Q\,dy \longrightarrow (3)$$

Adding (1) and (2) of RHS

$$\iint_{R_1} + \iint_{R_2} = \iint_{R} \longrightarrow (4)$$

$\therefore$ By (3) and (4) $\Rightarrow$ $\displaystyle\int_{BEAFB} Pdx + Qdy = \iint_{R} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Problems:

1) By using Green's theorem evaluate $\int_C (y - \sin x)\,dx + \cos x\,dy$ where C is the triangle in the xy-plane bounded by the lines $y = 0$, $x = \pi/2$ and $y = \frac{2x}{\pi}$

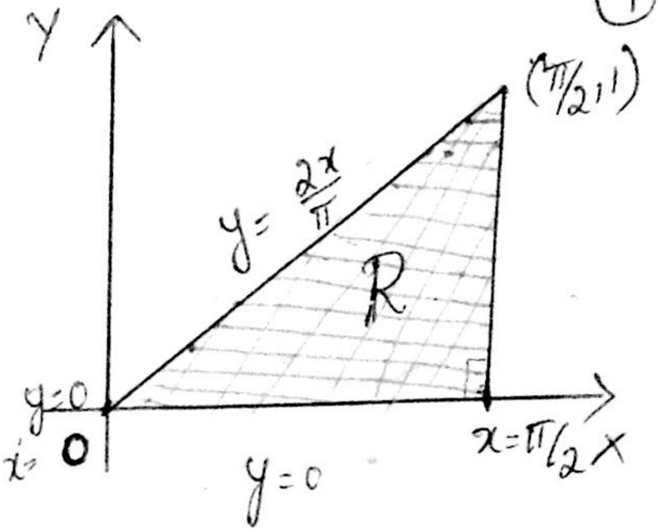
Solution: Here $P = y - \sin x$, $Q = \cos x$

$$\frac{\partial P}{\partial y} = 1 \quad , \quad \frac{\partial Q}{\partial x} = -\sin x$$

By Green's theorem we have,

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$$= \iint_R (-\sin x - 1)\,dx\,dy = -\iint_R (1 + \sin x)\,dx\,dy$$



$$= -\int_{x=0}^{\pi/2}\int_{y=0}^{\frac{2}{\pi}x} (1+\sin x)\, dy\, \cancel{dy}\, dx$$

$$= -\int_0^{\pi/2} \Big[(1+\sin x)\, y \Big]_0^{\frac{2}{\pi}x} dx$$

$$= -\int_0^{\pi/2} \Big[(1+\sin x)\left(\frac{2x}{\pi}\right) dx$$

$$= -\frac{2}{\pi}\int_0^{\pi/2} x(1+\sin x)\, dx$$

$$= -\frac{2}{\pi}\int_0^{\pi/2} (x + x\sin x)\, dx$$

$$= -\frac{2}{\pi}\left[\frac{x^2}{2} + \Big(x(-\cos x) + \sin x\,(1)\Big)\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[\frac{x^2}{2} - x\cos x + \sin x\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}\left[\left(\frac{1}{2}\,\frac{\pi^2}{4} - \frac{\pi}{2}\cancel{\cos\frac{\pi}{2}}^{\,0} + \sin\frac{\pi}{2}\right) - \left(0 - 0\cos 0 + \sin 0\right)\right]$$

$$= -\frac{2}{\pi}\left[\frac{\pi^2}{8} + 1\right]$$

$$\left[\frac{\cancel{2}^{\,1}}{\cancel{\pi}} \times \frac{\pi^{\cancel{2}}}{\cancel{8}_{4}} = \pi/4\right]$$

$$= -\left[\pi/4 + \frac{2}{\pi}\right]_{//}$$

2) Using Green's theorem evaluate $\int_C e^{-x}\sin y\,dx + e^{-x}\cos y\,dy$ where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,\pi/2)$ and $(0,\pi/2)$.

<u>Solution:</u> Here $P = e^{-x}\sin y$, $Q = e^{-x}\cos y$

$$\frac{\partial P}{\partial y} = e^{-x}\cos y, \quad \frac{\partial Q}{\partial x} = -e^{-x}\cos y$$

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{\pi}\int_{y=0}^{\pi/2} \left(2e^{-x}\cos y\right) dy\,dx$$